B.Sc-II Paper 3

Real Analysis Notes-9

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Theorems on continuity.

Th1. If f, g, be two functions continuous at a point c then the functions $f+g$, $f-g$, fg are also continuous at c if $g(c) \neq 0$, then f/g is also continuous at c.

Proof – Try by yourself.

Th2. A function f defined on an interval I is continuous at a point $c \in I$ iff for every sequence $\{c_n\}$ in I converging to c we have.

$$\lim_{n \to \infty} f(c_n) = f(c)$$

First let us suppose that the func f is continuous at a point $c \in I$, and $\{c_n\}$ is a sequence in I such that $\lim_{n \to \infty} c_n = c$

Since f is continuous at c, therefore for any given $\epsilon > 0$, a $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \text{ when}$$

$0 < |x - c| < \delta$

Again since $\lim_{n \to \infty} C_n = C$ — ①

therefore $\exists$ a positive integer m such that

$|C_n - C| < \delta, \forall n \geq m$ — ②

From ① putting $x = C_n$, we have

$|f(C_n) - f(C)| < \epsilon$ when $|C_n - C| < \delta$

$\Rightarrow |f(C_n) - f(c)| < \epsilon \ \forall n \geq m$ (using ②)

$\Rightarrow$ the sequence $\{f(C_n)\}$ converges to $f(C)$

$$\lim_{n \to \infty} f(C_n) = f(C)$$

Let us now suppose that f is not continuous at c, we shall now show that though $\exists$ a sequence $\{C_n\}$ in I converging to c yet the sequence $\{f(C_n)\}$ does not converge to $f(C)$.

Since f is not continuous at c, therefore there exists an $\epsilon > 0$ such that for every $\delta > 0$, $\exists$ an $x \in I$ such that

$|f(x) - f(c)| \geq \epsilon$ when $|x-c| < \delta$

$\therefore$ By taking $\delta = 1/n$, we find that for each positive integer n, there is a $c_n \in I$ such that

$|f(c_n) - f(c)| \geq \epsilon$ when $|c_n - c| < \frac{1}{n}$

Thus the sequence $\{f(c_n)\}$, does not converge to $f(c)$ while the sequence $\{c_n\}$ converges to c. //